GROUP THEORY-II

CORE PAPER -XIV

1 mark questions

1. What is the definition of an automorphism?
2. Define inner automorphism.
3. What is the automorphism group of a finite cyclic group?
4. State the application of factor groups to automorphism groups.
5. What is a characteristic subgroup?
6. Define the commutator subgroup and its properties.
7. What are the properties of external direct products?
8. How can the group of units modulo n be represented as an external direct product?
9. Define internal direct products.
10. State the Fundamental Theorem of finite abelian groups.
11. Define group actions.
12. What are stabilizers and kernels in group actions?
13. What is the permutation representation associated with a given group action?
14. State the application of group actions in Generalized Cayley's theorem.
15. What is the index theorem?
16. What are groups acting on themselves by conjugation?
17. Define class equation and its consequences.
18. What is conjugacy in Sn?
19. State Sylow's theorems and their consequences.
20. What is Cauchy's theorem?
21. An automorphism is a __________ map of a group that maps the group onto itself and preserves the group structure. (bijective)
22. An inner automorphism is induced by __________ by an element of the group. (conjugation)
23. A subgroup H of a group G is called characteristic if it is fixed under every __________ of G. (automorphism)
24. The inverse of a group homomorphism f: G → H is a homomorphism f^{-1}: H → G defined by $f^{-1}(h) = g$, where f(g) = h.
25. The trivial group {e} has no non-trivial automorphisms because any automorphism must map e to __________. (itself)
26. The commutator subgroup of a group G, denoted by [G,G], is the subgroup generated by all __________. ($[g,h] = g^{-1}h^{-1}gh$)
27. The Chinese Remainder Theorem for groups states that if G and H are finite groups with relatively prime orders, then G × H is isomorphic to the external direct product G ⊕ H.
28. The order of an element $(g_1,g_2,...,g_n)$ in the external direct product $G_1 \times G_2 \times ... \times G_n$ is $\mathrm{lcm}(\mathrm{ord}(g_1),\mathrm{ord}(g_2),...,\mathrm{ord}(g_n))$
29. A divisible abelian group is a group in which every element has a __________ for every positive integer k. (kth root)

30. The Fundamental Theorem of finite abelian groups states that every finite abelian group is isomorphic to a __________ of cyclic groups. (direct product)
31. A group action is a function that assigns to each element of a group G a permutation of a set X, such that the identity element of G is mapped to the identity permutation of X and the group operation of G is preserved by __________ of permutations. (composition)
32. The kernel of a group action is the set of elements of G that fix every element of X under the action.
33. The permutation representation associated with a given group action is a homomorphism from G to the __________ group on X. (symmetric)
34. The Generalized Cayley's theorem states that every group G is isomorphic to a subgroup of the __________ group on G, via the left regular representation. (symmetric)
35. The index of a subgroup H in a group G is the number of distinct __________ of H in G, denoted by [G:H]. (left cosets)
36. The center of a group G is the set of elements that __________ with every element of G, denoted by Z(G). (commute)
37. The class equation of a group G is a formula that expresses the order of G as a sum of the orders of its __________ classes.
38. The first Sylow theorem states that if p is a prime that divides the order of a finite group G, then G has a subgroup of order p^n for some $n \geq 0$.
39. A p-group is a group in which every element has order that is a power of __________. (p)
40. The second Sylow theorem states that if p is a prime that divides the order of a finite group G, then any two subgroups of G of order p^n are __________ by an element of G. (conjugate)
41. An ____________ of a group G is an isomorphism from G to itself.
42. An ____________ of a group G is an automorphism that is induced by conjugation by an element of G.
43. The set of all automorphisms of a group G, denoted Aut(G), forms a _________ under composition.
44. The automorphism group of a finite cyclic group of order n is isomorphic to the ____________ group Zn*.
45. The automorphism group of an infinite cyclic group is isomorphic to the ____________ group Z*.
46. The ____________ subgroup, denoted [G, G], of a group G is the subgroup generated by all commutators [a, b] = aba^-1b^-1, where a and b are elements of G.
47. The commutator subgroup [G, G] is always a ____________ subgroup of G.
48. A ____________ action of a group G on a set X is a function that assigns to each element $g \in G$ a permutation of X, such that the identity element of G is assigned the identity permutation.
49. The ____________ of a group action is the set of elements in X that are fixed by a particular element $g \in G$.

50. The ____________ of a group action is the set of elements in G that fix a particular element x ∈ X.
51. A group G acts on itself by ____________, where conjugation by an element g ∈ G maps each element x ∈ G to gxg^-1.
52. The ____________ equation relates the order of a group to the number of conjugacy classes and their sizes.

2 marks questions

1. Define an isomorphism.
2. What is the inverse of an automorphism?
3. State the Cayley's theorem.
4. Define a normal subgroup.
5. Give an example of a non-characteristic subgroup.
6. Define the direct sum of groups.
7. State the Chinese Remainder Theorem for groups.
8. What is the order of an element in a direct product?
9. Define the torsion subgroup of an abelian group.
10. State the classification theorem for finitely generated abelian groups.
11. Define the orbit of an element under a group action.
12. What is the orbit-stabilizer theorem?
13. Define the permutation group of a set.
14. State Burnside's lemma.
15. What is the cycle decomposition of a permutation?
16. Define the center of a group.
17. What is a conjugacy class?
18. State the first Sylow theorem.
19. What is a p-group?
20. State the second Sylow theorem.
21. What is the difference between an automorphism and an inner automorphism?
22. State the definition of a characteristic subgroup.
23. Can every subgroup of a group be a characteristic subgroup? Justify your answer.
24. Define the inverse of a group homomorphism.
25. Give an example of a group that has no non-trivial automorphisms.
26. Define the commutator subgroup of a group.
27. State the Chinese Remainder Theorem for groups.
28. What is the order of an element in an external direct product?
29. Define a divisible abelian group.
30. State the Fundamental Theorem of finite abelian groups.
31. Define a group action.
32. What is the kernel of a group action?
33. Define the permutation representation associated with a given group action.
34. State the Generalized Cayley's theorem.
35. What is the index of a subgroup?

36. Define the center of a group.
37. What is the class equation of a group?
38. State the first Sylow theorem.
39. Define a p-group.
40. State the second Sylow theorem.

6/7 marks questions

1. Define automorphism and inner automorphism. Discuss the differences between them with suitable examples.
2. Explain the concept of characteristic subgroups in a group. Give examples of characteristic subgroups and non-characteristic subgroups.
3. Can every subgroup of a group be a characteristic subgroup? Justify your answer with examples.
4. Define the inverse of a group homomorphism. Discuss its properties and give examples.
5. Give an example of a group that has no non-trivial automorphisms. Explain why such a group exists.
6. Define the commutator subgroup of a group. Discuss its properties and give examples.
7. State and prove the Chinese Remainder Theorem for groups. Give examples to illustrate its application.
8. What is the order of an element in an external direct product? Explain with examples.
9. Define a divisible abelian group. Discuss its properties and give examples.
10. State and prove the Fundamental Theorem of finite abelian groups. Give examples to illustrate its application.
11. Define a group action and discuss its properties. Give examples of group actions and their applications.
12. What is the kernel of a group action? Discuss its properties and give examples.
13. Define the permutation representation associated with a given group action. Discuss its properties and give examples.
14. State and prove the Generalized Cayley's theorem. Give examples to illustrate its application.
15. What is the index of a subgroup? Discuss its properties and give examples.
16. Define the center of a group and discuss its properties. Give examples to illustrate its application.
17. What is the class equation of a group? Discuss its properties and give examples.
18. State and prove the first Sylow theorem. Give examples to illustrate its application.
19. Define a p-group and discuss its properties. Give examples to illustrate its application.
20. State and prove the second Sylow theorem. Give examples to illustrate its application.