TOPOLOGY OF METRIC SPACES

CORE-9

1 mark questions

1. Define a metric space.
2. Give an example of a metric space.
3. State the definition of a Cauchy sequence.
4. Define a complete metric space.
5. Explain the concept of an open ball in a metric space.
6. Define a closed ball in a metric space.
7. What is a neighborhood of a point in a metric space?
8. Define an open set in a metric space.
9. Explain the concept of the interior of a set.
10. Define a limit point of a set in a metric space.
11. Define a closed set in a metric space.
12. State Cantor's theorem.
13. Define a metric space (X, d).
14. Give an example of a metric space.
15. State the Cauchy sequence criterion for metric spaces.
16. Define a complete metric space.
17. Write the formula for the open ball B(x, r) centered at x with radius r.
18. Write the formula for the closed ball $\bar{B}(x,r)$ centered at x with radius r.
19. Define a neighborhood N(x, ε) of a point x.
20. Define an open set in a metric space.
21. Write the notation for the interior of a set A.
22. Define a limit point of a set A.
23. Define a closed set in a metric space.
24. State Cantor's theorem using set cardinality.
25. Define a subspace of a metric space.
26. Explain what it means for a metric space to satisfy the first countability axiom.
27. State Baire's Category Theorem.
28. Define a subspace of a metric space.
29. State the first countability axiom for a topological space.
30. State Baire's Category Theorem.
31. Define a continuous mapping between metric spaces.
32. State an extension theorem for continuous functions.
33. Define uniform continuity of a function.
34. Define a homeomorphism between metric spaces.
35. Explain the concept of equivalent metrics in a metric space.
36. Define an isometry between metric spaces.
37. Explain what it means for a sequence of functions to uniformly converge.
38. Define continuity of a function f at a point x.
39. State an extension theorem for continuous functions.
40. Define uniform continuity of a function f.
41. Define a homeomorphism between metric spaces.
42. Define equivalent metrics on a metric space.

43. Define an isometry between metric spaces.
44. Write the definition of uniform convergence of a sequence of functions.
45. Define a contraction mapping.
46. State an application of contraction mappings.
47. Define a connected set in a metric space.
48. Define a locally connected set in a metric space.
49. Explain what it means for a set to be bounded in a metric space.
50. Define compactness of a set in a metric space.
51. State another characterization of compactness.
52. Explain the concept of continuous functions on compact spaces.
53. Define a contraction mapping on a metric space.
54. State a property of contraction mappings.
55. Define a connected set in a metric space.
56. Define a locally connected set in a metric space.
57. Write the definition of a bounded set in a metric space.
58. Define compactness of a set in a metric space.
59. State a property that characterizes compactness.
60. Define continuous functions on compact spaces.

2/3 marks questions

1. Prove that every convergent sequence in a metric space is a Cauchy sequence.
2. Given a metric space (X, d), show that the Euclidean metric on ℝ^n satisfies the triangle inequality.
3. Prove that the intersection of any collection of open sets in a metric space is an open set.
4. Using the definition of limit points, show that a point x is a limit point of a set A if and only if every open ball centered at x contains a point of A distinct from x.
5. Prove that the closure of a set A is the union of A and its set of limit points.
6. Given a metric space (X, d), show that the empty set ∅ is open and X is closed.
7. Prove that a set A in a metric space is compact if and only if every open cover of A has a finite subcover.
8. Define a metric space (X, d) and give an example.
9. Prove that every convergent sequence in a metric space is a Cauchy sequence.
10. Define a complete metric space and provide an example.
11. Prove that a closed subset of a complete metric space is complete.
12. Explain the concept of an open ball B(x, r) and its relationship with neighborhoods.
13. Prove that the intersection of a finite number of open sets is an open set.
14. Define a limit point of a set A in a metric space.
15. Prove that a set A is closed if and only if it contains all its limit points.
16. Define a bounded set in a metric space.
17. Prove that a closed and bounded subset of Euclidean space is compact.

18. Prove that a subspace of a separable metric space is separable.
19. Show that the countable union of countable sets is countable.
20. Prove that if X is a complete metric space, then Baire's Category Theorem holds for X.
21. Define a subspace of a metric space and provide an example.
22. Prove that a subset of a metric space is a subspace if and only if it is closed under scalar multiplication and vector addition.
23. Explain the first countability axiom and how it is related to sequences.
24. State and prove the result that every sequence in a metric space has a countable set of limit points.
25. State and prove Baire's Category Theorem.
26. Prove that the composition of continuous functions is continuous.
27. Show that if a sequence of continuous functions converges uniformly to a limit function, then the limit function is also continuous.
28. Given a continuous function $f: [a, b] \to \mathbb{R}$, prove that f is uniformly continuous on $[a, b]$.
29. Prove that a continuous bijection from a compact metric space to a Hausdorff space is a homeomorphism.
30. Show that if a sequence of functions $\{f_n\}$ uniformly converges to f, then the limit function f is bounded if the functions $\{f_n\}$ are bounded.
31. Define the continuity of a function at a point using ε-δ definition.
32. Prove that a composition of continuous functions is continuous.
33. Define uniform continuity of a function and provide an example.
34. Prove that a uniformly continuous function maps Cauchy sequences to Cauchy sequences.
35. Define homeomorphism between metric spaces and explain its topological significance.
36. State the Bolzano-Weierstrass theorem and prove it.
37. Define the pointwise and uniform convergence of a sequence of functions.
38. Prove that the uniform limit of a sequence of continuous functions is itself continuous.
39. Prove the Banach Fixed-Point Theorem for contraction mappings.
40. Show that the image of a connected set under a continuous function is connected.
41. Prove that the continuous image of a compact set is compact.
42. Using the Heine-Borel Theorem, show that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
43. Prove that a compact subset of a metric space is sequentially compact.
44. Define a contraction mapping and its properties.
45. State the Banach Fixed-Point Theorem and provide its proof.

46. Define a connected set in a metric space and give an example.
47. Prove that the continuous image of a connected set is connected.
48. Define a compact set in a metric space.
49. Prove that a closed subset of a compact set is compact.
50. Define the Heine-Borel Theorem and prove it.
51. Define the concept of compactness in a topological space and its relation to metric spaces.

6/7 marks questions

1. Define a metric space (X, d) and prove that the distance function d satisfies the properties of a metric.
2. Prove that a Cauchy sequence in a metric space is bounded.
3. Define the completion of a metric space and show that it is unique up to isometric isomorphism.
4. Prove that the union of a finite number of closed sets in a metric space is closed.
5. Define a compact set in a metric space and prove that every closed interval [a, b] in R is compact.
6. Prove that a compact subset of a metric space is closed and bounded.
7. Define the concept of sequentially compactness in a metric space and provide an example of a sequentially compact set.
8. Prove that a sequentially compact set in a metric space is also bounded.
9. Define the concept of totally boundedness in a metric space and prove that every totally bounded set is bounded.
10. Prove that a closed subset of a totally bounded set in a metric space is totally bounded.
11. Define a subspace of a metric space (X, d) and prove that the subspace inherits the metric from X.
12. Prove that a countable union of countable sets is countable.
13. Define a nowhere dense set in a metric space and prove that the interior of the closure of a nowhere dense set is empty.
14. Prove that the intersection of a countable collection of dense open sets in a metric space is dense.
15. Define a meager set in a metric space and state the Cantor Baire Theorem.
16. Prove that if a metric space (X, d) is a complete metric space, then every closed subspace of X is also complete.
17. Define a Banach space and prove that a finite-dimensional subspace of a Banach space is closed.
18. Prove that every open covering of a sequentially compact metric space has a finite subcovering.

19. Define the concept of a perfect set in a metric space and provide an example.
20. Prove that every nonempty perfect set in a metric space is uncountable.
21. Define the concept of a uniform limit of a sequence of functions and prove that the uniform limit of continuous functions is continuous.
22. Prove that a uniformly convergent sequence of functions can be integrated term-wise over a closed and bounded interval.
23. Define the concept of a uniformly equicontinuous family of functions and provide an example.
24. Prove the Arzelà–Ascoli Theorem for uniformly equicontinuous and pointwise bounded families of functions.
25. Define the concept of a homeomorphism between metric spaces and prove that a homeomorphism preserves compactness.
26. Prove that a continuous bijection from a compact metric space to a Hausdorff space is a homeomorphism.
27. Define the concept of a complete metric space and prove that a compact metric space is complete.
28. Prove that a compact subset of a metric space is totally bounded.
29. Define the concept of a locally compact metric space and provide an example.
30. Prove that a locally compact metric space has an open neighborhood basis consisting of compact sets.
31. Define the concept of a contraction mapping and state the Contraction Mapping Theorem.
32. Prove the Contraction Mapping Theorem: Every contraction mapping on a complete metric space has a unique fixed point.
33. Define the concept of a connected set in a metric space and prove that the continuous image of a connected set is connected.
34. Prove that a connected subset of a metric space cannot be expressed as the union of two disjoint nonempty open sets.
35. Define the concept of a compact metric space and prove that every closed subset of a compact metric space is compact.
36. Prove that a continuous function on a compact metric space is uniformly continuous.
37. Define the concept of compactness in a topological space and prove that a continuous image of a compact space is compact.
38. Prove that the product of two compact metric spaces is compact.
39. Define the concept of a locally compact metric space and provide an example.
40. Prove that a compact metric space is sequentially compact.